Bilad Alrafidain University College

Electric Power Techniques Engineering Department

Control Systems Analysis

Fourth Stage

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Lecture Eleven

Root Locus Method

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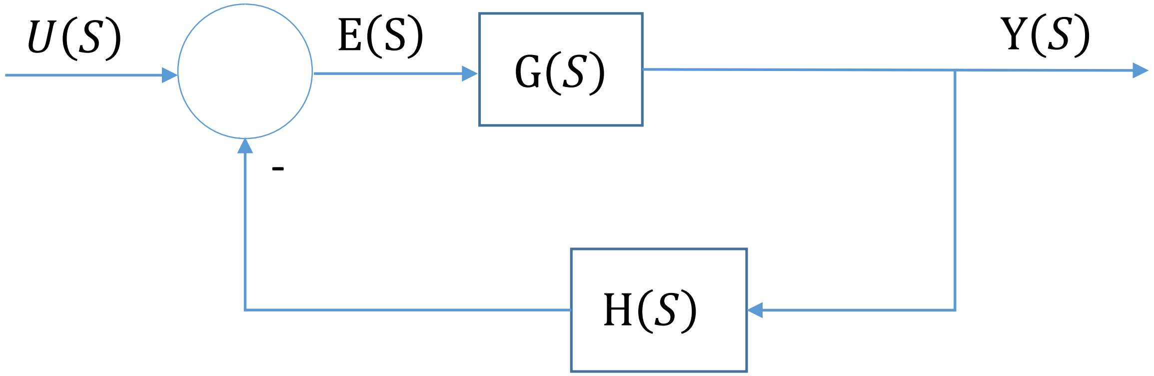
Root Locus Method

$$\frac{Y(S)}{U(S)} = \frac{G(S)}{1 + G(S)H(S)}$$ Transfer function



$$P(S) = 1 + G(S)H(S) = 0$$ Characteristic equation

$$G(S)H(S) = -1$$

In many cases, $G(s)H(s)$ involves a gain parameter K;

Then the root loci for the system are the loci of the closed loop poles as the gain K is varied from zero to infinity, $0 \leq K \leq \infty$.

To begin sketching the root loci of a system by the root-locus method the location of the poles and zeros of $G(s)H(s)$ are needed.

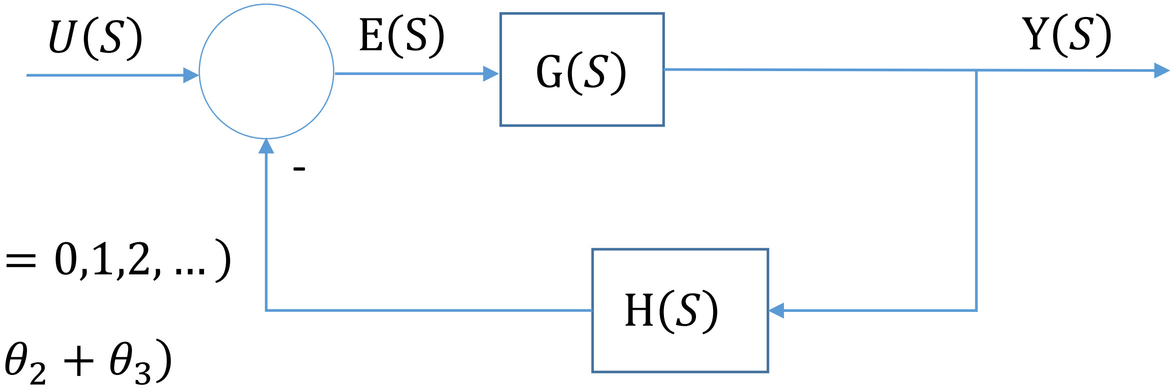
Root Locus Method

Ones the poles and zeros of open loop transfer function $G(s)H(s)$ is determined in s plane. The angles of the complex quantities originating from the open-loop poles $\theta_1, \theta_2 \ldots \theta_n$ and open-loop zeros $\varphi_1, \varphi_2, \ldots \varphi_m$ to the test point s_1 are measured in the counter clockwise direction.

If the test point s_1 in the root loci then the magnitude of characteristic equation is: $|G(S)H(S)| = 1$

Root Locus Method

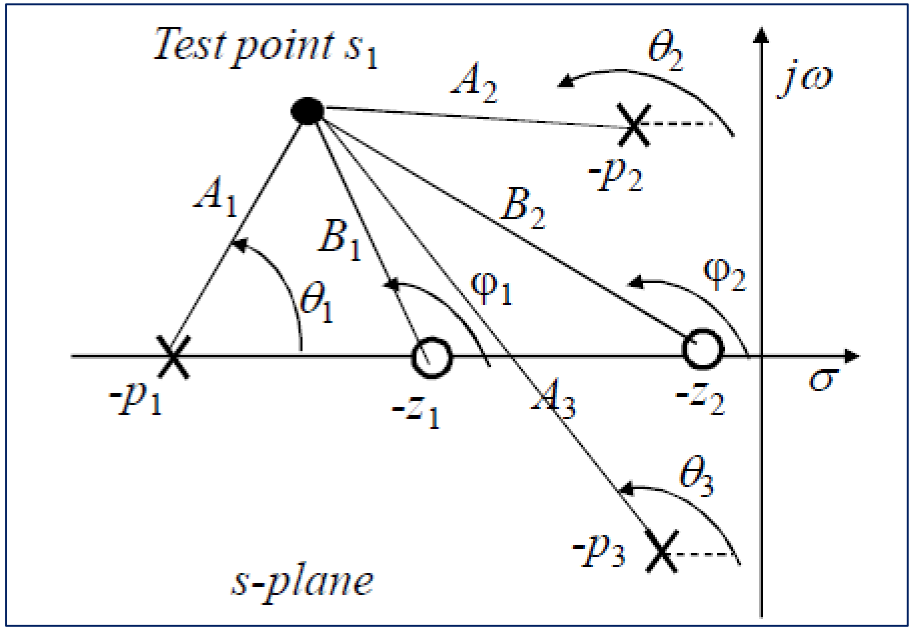
$$G(S)H(S) = \frac{K(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)(s+p_3)}$$



Angle Condition: $\angle KG(S)H(S) = \pm 180^{\circ}(2K+1), (K = 0,1,2,\dots)$

$$\angle KG(S)H(S) = (\varphi_1 + \varphi_2) - (\theta_1 + \theta_2 + \theta_3)$$

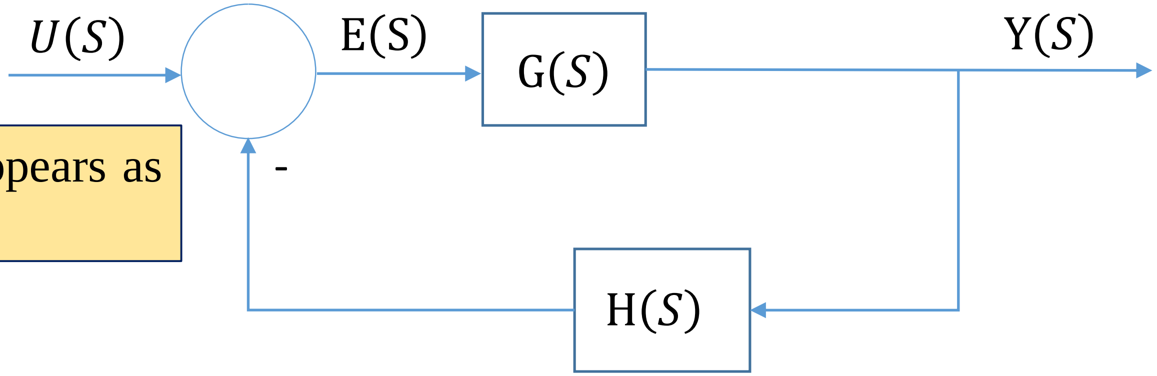
Magnitude Condition: $|KG(S)H(S)| = 1 \Rightarrow \frac{KB_1B_2}{A_1A_2A_3} = 1$



A_1, A_2 and A_3 are the magnitudes of the complex quantities $s+p_1$, $s+p_2$ and $s+p_3$ respectively at test point s_1. B_1, and B_2 are the magnitudes of the complex quantities $s+z_1$ and $s+z_2$ respectively at test point s_1.

Root Locus Method

$P(S) = 1 + G(S)H(S) = 0$ Characteristic equation



Then rearrange this equation so that the parameter of interest appears as the multiplying factor in the form:

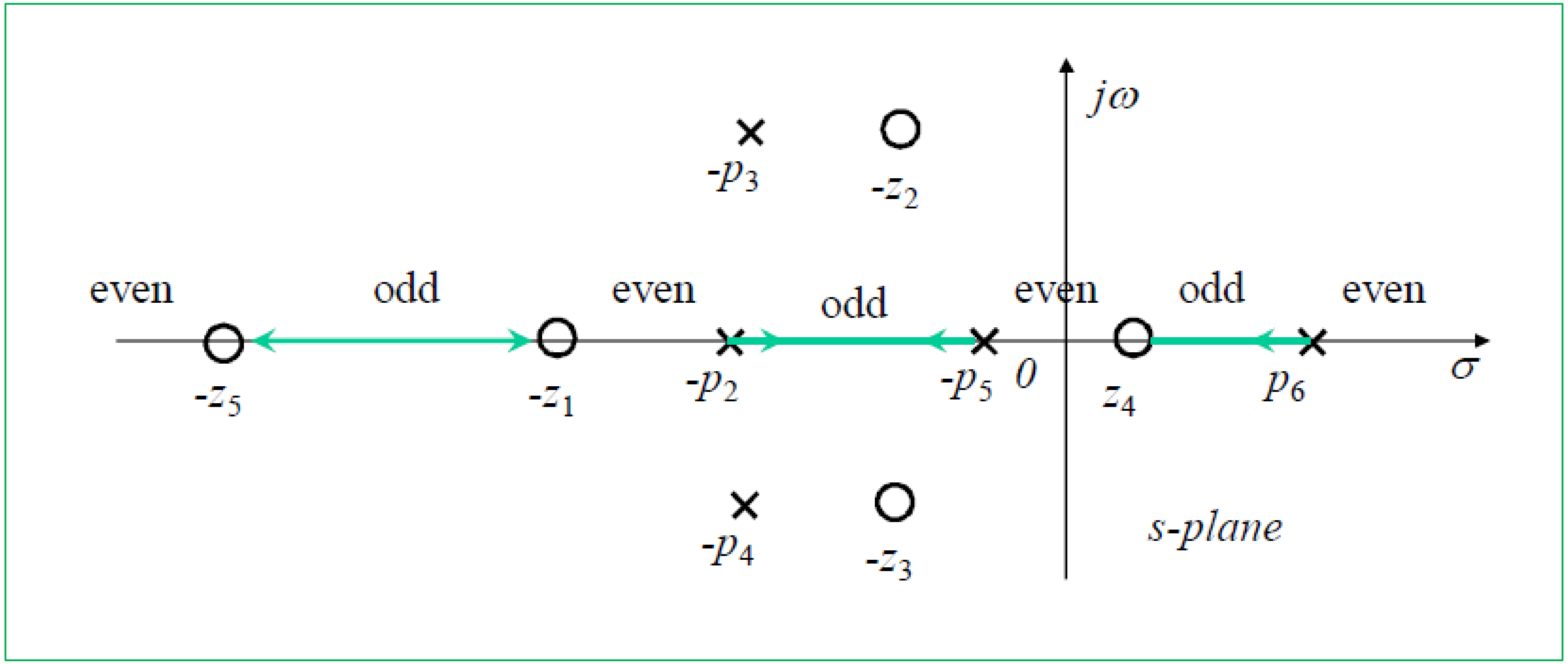
$$P(S) = 1 + G(S)H(S) = 0 \Rightarrow P(S) = 1 + \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{S^N(s+p_1)(s+p_2)\ldots(s+p_{n-N})} = 0$$

General Rules for Constructing Root Loci

1. Locate the poles and zeros of $G(s)H(s)$ on the s plane : The root-locus branches start at the poles of $G(s)H(s)$ and end at the zeros of $G(s)H(s)$ as K increases from zero to infinity, $0 \leq K \leq \infty$, where the poles and zeros include both those in the finite s plane and those at infinity.

2. Determine the root loci on the real axis : Root loci on the real axis are determined by open-loop transfer function $G(s)H(s)$ poles and zeros lying on it. In constructing the root loci on the real axis, choose a test point s_1 on it. If the total number of real poles and real zeros to the right of this test point is odd, then this point lies on a root locus. If the open-loop poles and open-loop zeros are simple poles and simple zeros, then the root locus and its complement form alternate segments along the real axis.

General Rules for Constructing Root Loci



General Rules for Constructing Root Loci

3. Determine the asymptotes of root loci : If the test point s_1 is located far from the origin, then the angle of each complex quantity may be considered the same. One open-loop zero and one open-loop pole then cancel the effects of the other. Therefore the root loci for very large values of s must be asymptotic to straight lines whose angles (slopes) are given by

$$\emptyset_j = \frac{\pm 180^\circ(2K+1)}{n-m}, K = 0,1,2,\ldots$$

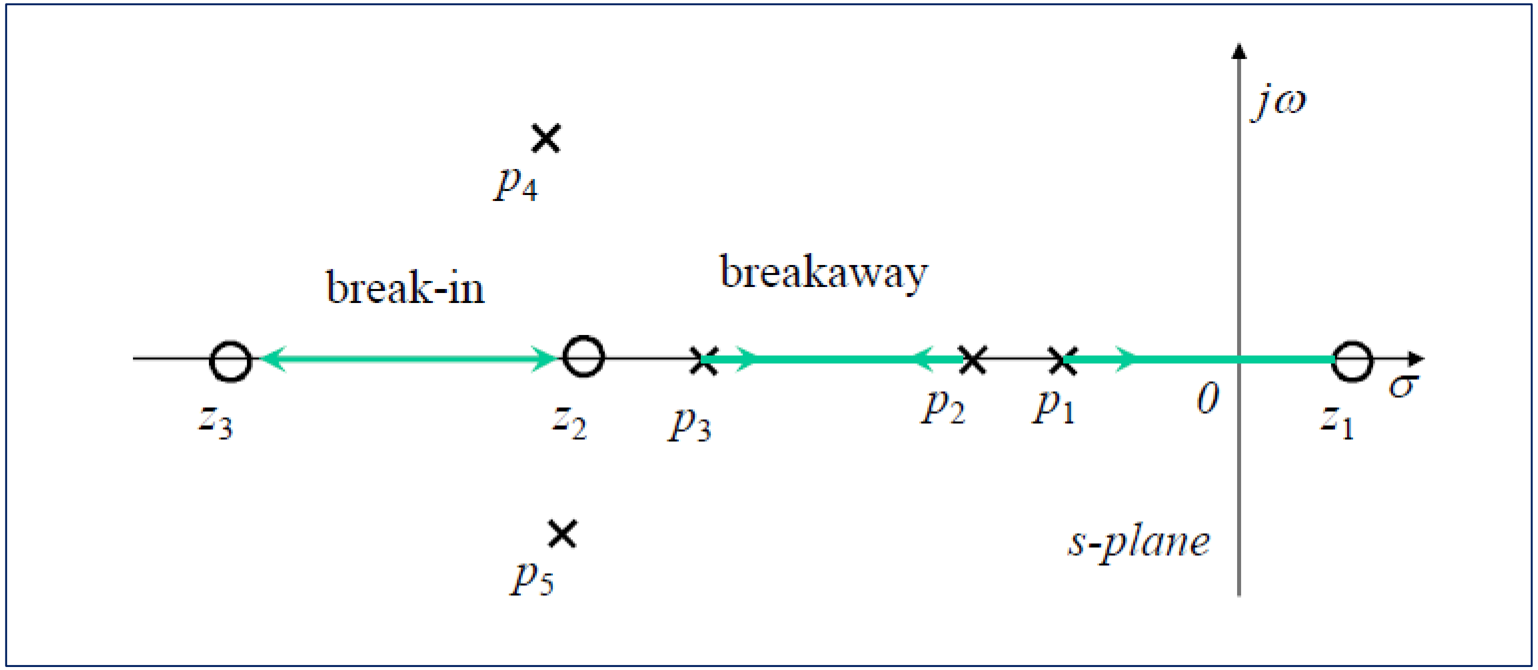
where n and m are respectively the number of finite poles and zeros of open loop transfer function $G(s)H(s)$.

Here, $k = 0$ corresponds to the asymptotes with the smallest angle with the real axis. Although k assumes an infinite number of values, as k is increased the angle repeats itself, and the number of distinct asymptotes is $n - m$.

All the asymptotes intersect on the real axis at: $$\sigma = -\frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m}$$

General Rules for Constructing Root Loci

4. Find the breakaway and break-in points : The breakaway point corresponds to a point in the s plane where multiple roots of the characteristic equation occur. A break-in point exists where a pair of root-locus branches coalesces as K is increased.

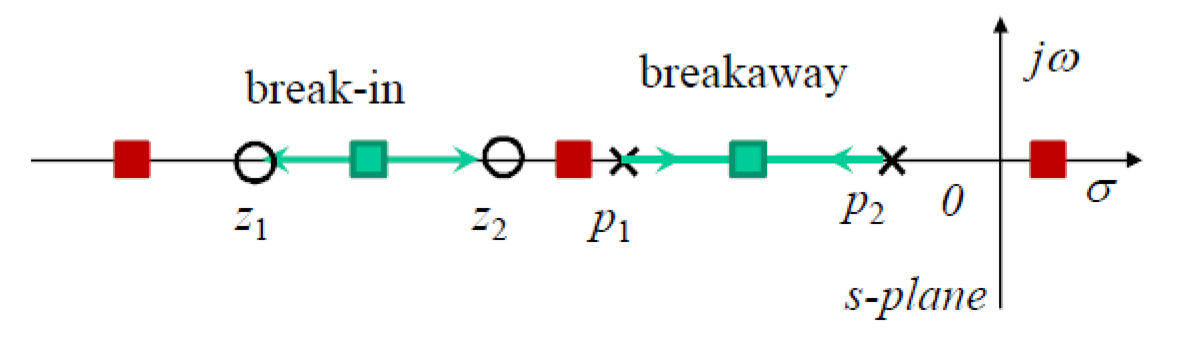


General Rules for Constructing Root Loci

Suppose that the characteristic equation is given by $P(S) = 1 + KG(S)H(S) = 1 + K\frac{A(S)}{B(S)} = 0$

$K = -\frac{B(S)}{A(S)}$ The breakaway and break-in points can be determined from the roots of

$$\frac{dK}{ds} = -\frac{d}{ds}\left[\frac{B(S)}{A(S)}\right] = 0 \Rightarrow \frac{A(S)\dot{B}(S) - B(S)\dot{A}(S)}{A^2(S)} = 0$$



where the prime indicates differentiation with respect to *s*. However not all roots are breakaway or break-in points. If a real root lies on the root-locus portion of the real axis, then it is an actual breakaway or break-in point. If a real root is not on the root-locus portion of the real axis, then this root corresponds to neither a breakaway point nor a break-in point.

General Rules for Constructing Root Loci

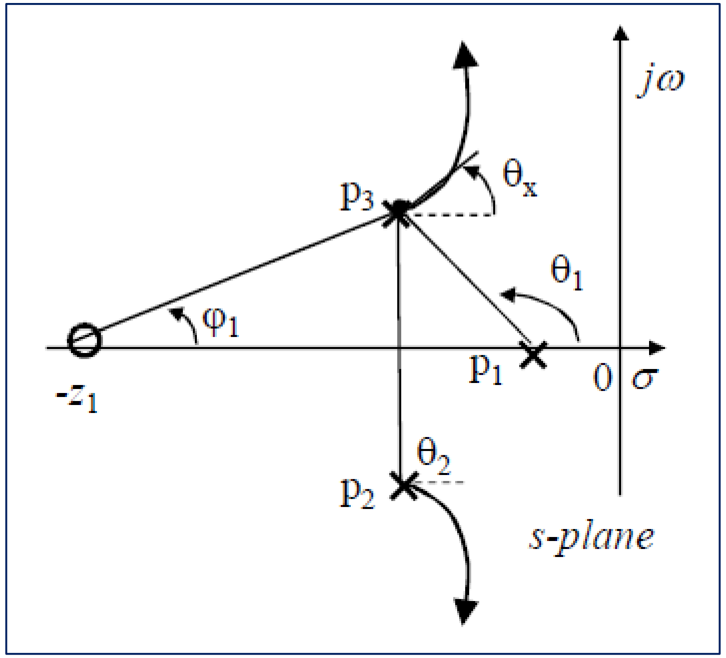
5. Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero) : The angle of departure (or angle of arrival) of the root locus from a complex pole (or at a complex zero) can be found by subtracting from 180" the sum of all the angles of vectors from all other poles and zeros to the complex pole (or complex zero) in question, with appropriate signs included.

Angle of departure from a complex pole

$$\theta_X = 180^\circ - \sum_{j=1}^{n-1} \theta_j + \sum_{i=1}^{m} \varphi_i$$

Angle of arrival at a complex zero

$$\theta_X = -180^\circ - \sum_{i=1}^{m} \varphi_i + \sum_{j=1}^{n-1} \theta_j$$



General Rules for Constructing Root Loci

6. Find the points where the root loci may cross the imaginary axis : The points where the root loci intersect the $j\omega$ axis can be found easily by

(*a*) use of Routh's stability criterion $P(S) = 1 + G(S)H(S) = 0$

or

(*b*) letting $s = j\omega$ in the characteristic equation, equating both the real part and the imaginary part to zero, and solving for ω and K.

$$P(S)\Big|_{s=j\omega} = 1 + G(j\omega)H(j\omega) = 0$$

$$\mathrm{Re}[P(j\omega)] = 0 \Rightarrow Re[G(j\omega)H(j\omega)] = -1$$

$$Im[P(j\omega)] = 0 \Rightarrow Im\big[[G(j\omega)H(j\omega)] = 0\big]$$

The values of ω thus found give the frequencies at which root loci cross the imaginary axis. The K value corresponding to each crossing frequency gives the gain at the crossing point.

General Rules for Constructing Root Loci

7. Taking a series of test points in the broad neighbourhood of the origin, $s = 0$, of the s plane, sketch the root loci : The most important part of the root loci is on neither the real axis nor the asymptotes, but the part in the broad neighbourhood of the $j\omega$ axis and the origin. The shape of the root loci in this important region in the s plane must be obtained with reasonable accuracy

8. Draw the root loci