Bilad Alrafidain University College

Electric Power Techniques Engineering Department

Control Systems Analysis

Fourth Stage

Academic Year 2020 - 2021

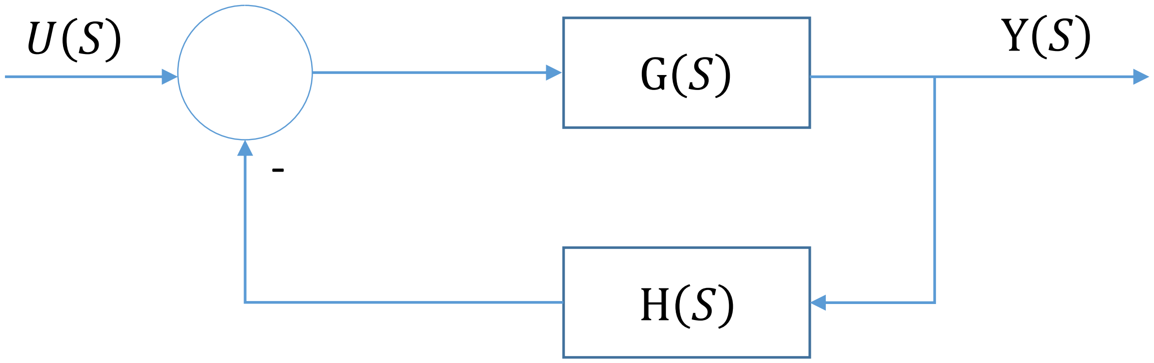
Lecture Eight

Stability Analysis

Assistant Lecturer. Ibrahim Ismail

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

$$\frac{Y(S)}{U(S)} = \frac{G(S)}{1 + G(S)H(S)}$$



$$P(S) = 1 + G(S)H(S) = 0$$

$$P(S) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

$$\dot{x} = Ax + Bu \qquad P(s) = |sI - A| = 0$$

$$y = Cx + Du \qquad P(S) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Routh's stability criterion tells us whether or not there are unstable roots in a characteristic polynomial equation, P(s), or eigenvalues of characteristic matrix without actually solving for them. This stability criterion applies to polynomials with only a finite number of terms. When the criterion is applied to a control system, information about absolute stability can be obtained directly from the coefficients of the characteristic equation.

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

The procedure in Routh's stability criterion is as follows:

1. Write the polynomial in s in the following form: $P(S) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$

where the coefficients are real quantities. We assume that $an \neq 0$; that is, any zero root has been removed.

2. If any of the coefficients are zero or negative in the presence of at least one positive coefficient, there is a root or roots that are imaginary or that have positive real parts. Therefore, in such a case, **the system is not stable**. If we are interested in only the absolute stability, there is no need to follow the procedure further. In order to go further step all the coefficients of P(s) must be positive. **The necessary but not sufficient condition for stability is that the coefficients of the characteristic equation all be present and all have a positive sign.**

$$P(S) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

3. Routh-Hurwitz stability criterion consists two steps:

a) Constructing of Routh-Hurwitz array

b) Applying the necessary and sufficient condition:

All roots of Equation Characteristic $P(s)$ lie in the left-half s plane is that all the coefficients of Equation $P(s)$ be positive and all terms in the first column of **Routh-Hurwitz array** have positive signs.

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

3. a) Constructing of Routh-Hurwitz array: The first two rows can be obtained directly from the given polynomial. The remaining terms are obtained from these.

Routh-Hurwitz array: $P(S) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$

$$
\begin{array}{c|ccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots
\end{array}
$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix},$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \quad b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}, \ldots$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}, \quad c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}, \ldots$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}, \quad d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}, \ldots$$

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

For n order $P(s)$ the total number of rows is $n+1$.

Routh-Hurwitz array: $P(S) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$

$$n+1 \left\{ \begin{array}{cccccc} s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\ s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\ s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\ s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\ s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s^2 & e_1 & e_2 & & & \\ s^1 & f_1 & & & & \\ s^0 & g_1 & & & & \end{array} \right.$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix},$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}, \ldots$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1},$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}, \ldots$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}, d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}, \ldots$$

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

3. b) Applying the necessary and sufficient condition: All roots of Equation $P(s)$ lie in the left-half of s-plane is that all the coefficients of Equation $P(s)$ be positive and all terms in the first column of **Routh-Hurwitz array** have positive signs.

Routh-Hurwitz array:

s^n	a_0	a_2	a_4	a_6	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
s^{n-4}	d_1	d_2	d_3	d_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^2	e_1	e_2			
s^1	f_1				
s^0	g_1				

The necessary and sufficient condition of the stability is :

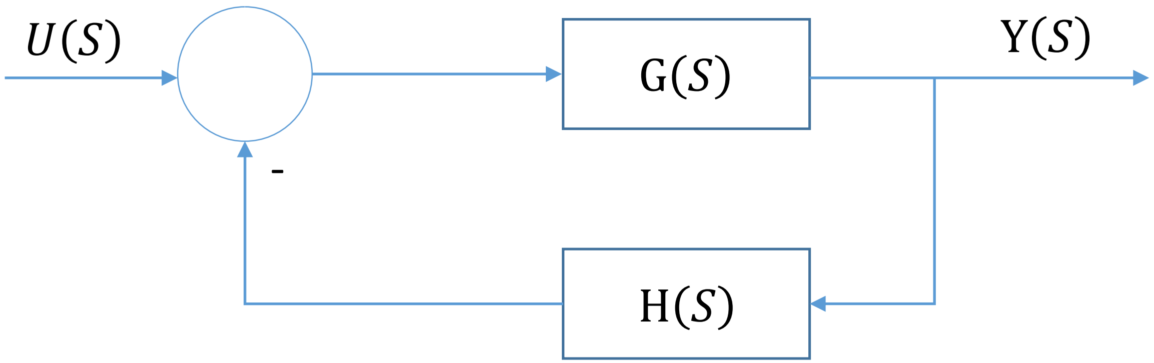
b.1) all the coefficients of Equation $P(s)$ be positive **and**

$$P(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

$$a_i > 0 \quad (i = 0,1,2,\ldots,n)$$

b.2) all terms in the first column of Routh-Hurwitz array have positive signs.

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion



$$P(S) = 1 + G(S)H(S) = 0$$

It should be noted that the exact values of the terms in the first column need not be known; instead, only the signs are needed.

Routh-Hurwitz array:

s^8	4	
s^7	3	
s^6	1	1. pole on RHP of s
s^5	-2	
s^4	4	2. pole on RHP of s
s^3	3	3. pole on RHP of s
s^2	-6	
s^1	3	4. pole on RHP of s
s^0	1	

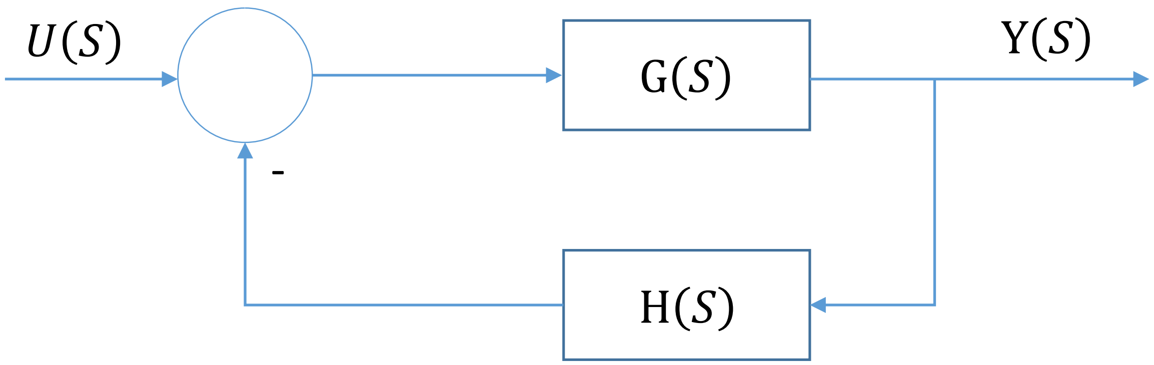
Routh's stability criterion states that: the number of roots of Characteristic Equation $P(s)$ with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array.

The system has four poles on the RHS of s-plane. Hence, the system is unstable.

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

Example 1. Let us apply Routh's stability criterion to the following system:

$$\frac{Y(S)}{U(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{G(s)}{s^4+2s^3+3s^2+4s+5}$$



$$P(S) = 1 + G(S)H(S) = 0$$

$$P(S) = s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

$$a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4, a_4 = 5$$

where all the coefficients are positive numbers. Thus the first **necessary** and **sufficient** condition is satisfied. But we need apply the second condition using **Routh-Hurwitz array.**

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

$$\frac{Y(S)}{U(S)} = \frac{G(S)}{1 + G(S)H(S)} = \frac{G(s)}{s^4 + 2s^3 + 3s^2 + 4s + 5}$$

$$P(S) = s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

$$a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4, a_4 = 5$$

Routh-Hurwitz array:

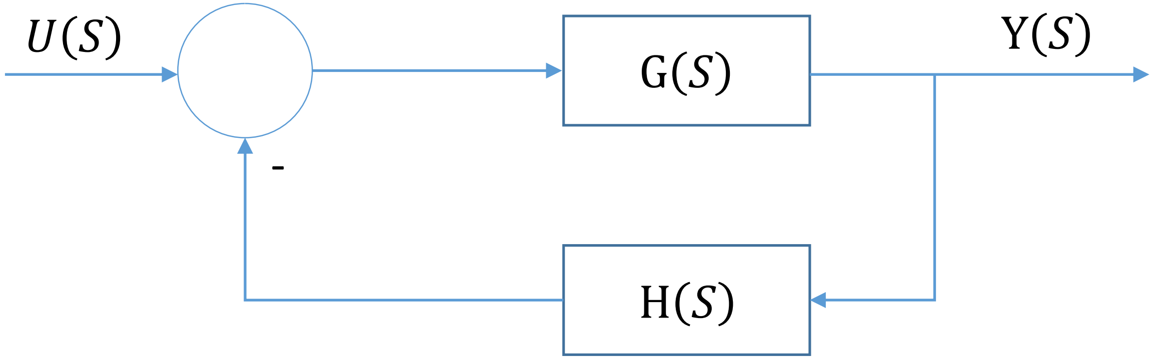
s^4	$a_0 = 1$	$a_2 = 3$	$a_4 = 5$
s^3	$a_1 = 2$	$a_3 = 4$	$a_5 = 0$
s^2	$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = 1$	$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} = 5$	$b_3 = 0$
s^1	$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1} = -6$	$c_2 = \frac{b_1 a_5 - b_3 a_1}{b_1} = 0$	$c_3 = 0$
s^0	$d_1 = \frac{c_1 b_2 - c_2 b_1}{c_1} = 5$		

In this example, the number of changes in sign of the coefficients in the first column is 2. This means that there are two roots with positive real part of *s*-plane. Hence the system is unstable.

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

Example 2. Let us apply Routh's stability criterion to the following system:

$$\frac{Y(S)}{U(S)} = \frac{G(S)}{1 + G(S)H(S)} = \frac{G(s)}{s(s^2 + s + 1)(s + 2) + 1}$$



$$P(S) = 1 + G(S)H(S) = 0$$

$$P(S) = s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$$

$$a_0 = 1, a_1 = 3, a_2 = 3, a_3 = 2, a_4 = 1$$

where all the coefficients are positive numbers. Thus the first **necessary** and **sufficient** condition is satisfied. But we need apply the second condition using **Routh-Hurwitz array.**

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

$$\frac{Y(S)}{U(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{G(s)}{s(s^2+s+1)(s+2)+1}$$

$$P(S) = s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$$

$$a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4, a_4 = 5$$

Routh-Hurwitz array:

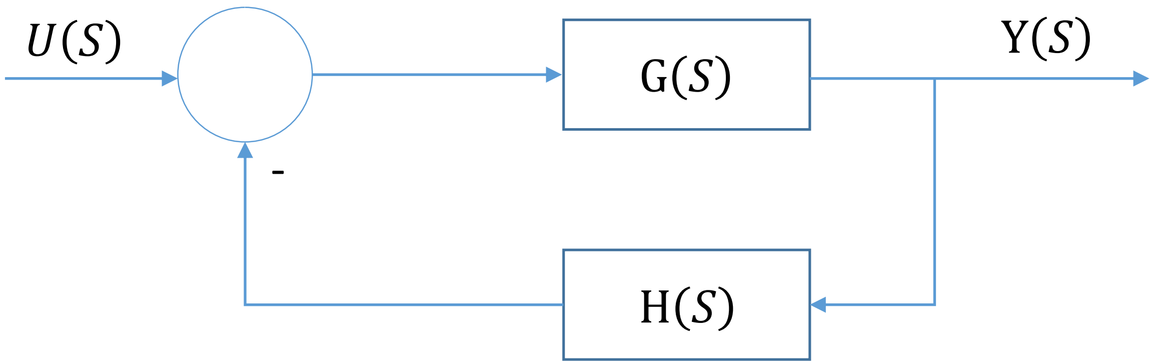
s^4	$a_0 = 1$	$a_2 = 3$	$a_4 = 1$
s^3	$a_1 = 3$	$a_3 = 2$	$a_5 = 0$
s^2	$b_1 = \frac{a_1a_2 - a_0a_3}{a_1} = 2.33$	$b_2 = \frac{a_1a_4 - a_0a_5}{a_1} = 1$	$b_3 = 0$
s^1	$c_1 = \frac{b_1a_3 - b_2a_1}{b_1} = 0.714$	$c_2 = \frac{b_1a_5 - b_3a_1}{b_1} = 0$	$c_3 = 0$
s^0	$d_1 = \frac{c_1b_2 - c_2b_1}{c_1} = 1$		

There is no changes in sign of the coefficients in the first column of **Routh-Hurwitz array. Hence the system is stable.**

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

Example 3. Discuses the stability of the following characteristic equations. Which characteristic equation stability can be determined by applying Routh- Hurwitz stability criterion:

$$\frac{Y(S)}{U(S)} = \frac{G(S)}{1 + G(S)H(S)}$$



1. $P_1(s) = s^5 + 5s^4 + 8s^2 + 20s + 35$

2. $P_2(s) = s^5 + 5s^4 + 4s^3 + 8s^2 - 10s + 15$

3. $P_3(s) = s^5 + 2s^4 + s^3 + 3s^2 + 5s + 10$

4. $P_4(s) = -s^4 - 7s^3 - 4s^2 - 3s - 5$

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

1. $P_1(s) = s^5 + 5s^4 + 8s^2 + 20s + 35$

Since the coefficients for s^3 is zero, there is a root or roots that has imaginary or that have positive real parts. Thus the system is unstable and does not need to apply Routh-Hurwitz stability criterion.

In fact this can be seen by finding the roots of the given characteristic equation from Matlab:

>> roots([1 5 0 8 20 35])

----

s_1= -5.2014
s_2= 1.0942 + 1.4973i
s_3= 1.0942 - 1.4973i
s_4= -0.9935 + 0.9847i
s_5= -0.9935 - 0.9847i

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

$$2.\ P_2(s) = s^5 + 5s^4 + 4s^3 + 8s^2 - 10s + 15$$

Since the coefficients for s has negative sign , there is a root or roots that has imaginary or that have positive real parts. Thus the system is unstable and does not need to apply Routh-Hurwitz stability criterion.

However this can be seen from the roots of the given characteristic equation from Matlab:

>> roots([1 5 4 8 -10 15])

----

s_1= -4.6419
s_2= -0.8114 + 1.6721i
s_3= -0.8114 - 1.6721i
s_4= 0.6323 + 0.7319i
s_5= 0.6323 - 0.7319i

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

$$3.\ P_3(s) = s^5 + 2s^4 + s^3 + 3s^2 + 5s + 10$$

Non of the coefficients of the characteristic equation are zero or negative. Thus the system can be stable or unstable. Thus Routh-Hurwitz stability criterion can be applied to see whether it is stable or not.

$$4.\ P_4(s) = -s^4 - 7s^3 - 4s^2 - 3s - 5$$

Taking negative parenthesis leads:

$$P_4(s) = -(s^4 + 7s^3 + 4s^2 + 3s + 5)$$

Again non of the coefficients of the characteristic equation are zero or negative. Thus Routh-Hurwitz stability criterion can be applied to see whether it is stable or not.

Stability of Linear Feedback Systems by Routh-Hurwitz Stability Criterion

Example 4. Discuses the stability of the following characteristic Matrix.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$P(s) = (SI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix} = \begin{bmatrix} s+1 & 1 \\ -6.5 & 0 \end{bmatrix}$$

$$P(s) = s^2 + s + 6.5$$

```
>> eig([-1 -1;6.5 0])
------------------------
```

s_1= -0.5 + 2.5i
s_2= -0.5 - 2.5i

In this example, two roots are complex (roots with imaginary parts). Hence the system is unstable.