Series and Parallel ac Circuits

SERIES AC CIRCUITS

1.1. IMPEDANCE AND THE PHASOR DIAGRAM

1. Resistive Elements

For the purely resistive circuit in Fig. 1, that v and i were in phase, and the magnitude

$$I_m = \frac{V_m}{R} \quad \text{or} \quad V_m = I_m R$$

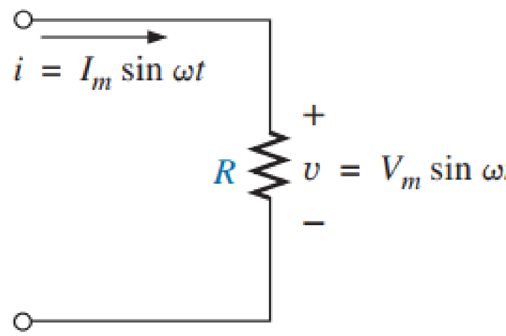

FIG. 1 Resistive ac circuit.

In phasor form,

$$v = V_m \sin \omega t \Rightarrow \mathbf{V} = V \angle 0^\circ$$

where $V = 0.707 V_m$.

Applying Ohm's law and using phasor algebra, we have

$$\mathbf{I} = \frac{V \angle 0^\circ}{R \angle \theta_R} = \frac{V}{R} \angle 0^\circ - \theta_R$$

Since i and v are in phase, the angle associated with i also must be 0°. To satisfy this condition, θ_R must equal 0°. Substituting $\theta_R = 0°$, we find

$$\mathbf{I} = \frac{V \angle 0^\circ}{R \angle 0^\circ} = \frac{V}{R} \angle 0^\circ - 0^\circ = \frac{V}{R} \angle 0^\circ$$

So that in the time domain,

$$i = \sqrt{2}\left(\frac{V}{R}\right) \sin \omega t$$

We use the fact that $\theta_R = 0°$ in the following polar format to ensure the proper phase relationship between the voltage and current of a resistor:

$$\boxed{\mathbf{Z}_R = R \angle 0°}$$

EXAMPLE 1 Using complex algebra, find the current i for the circuit in Fig.2. Sketch the waveforms and the phasor diagram of v and i.

Solution: Note Fig.3 and Fig. 4:

$$v = 100 \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = 70.71 \text{ V} \angle 0°$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_R} = \frac{V \angle \theta}{R \angle 0°} = \frac{70.71 \text{ V} \angle 0°}{5 \ \Omega \angle 0°} = 14.14 \text{ A} \angle 0°$$

and

$$i = \sqrt{2}(14.14) \sin \omega t = \mathbf{20 \sin \omega t}$$

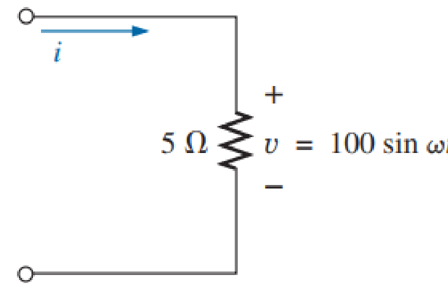

FIG. 2 Example 1.

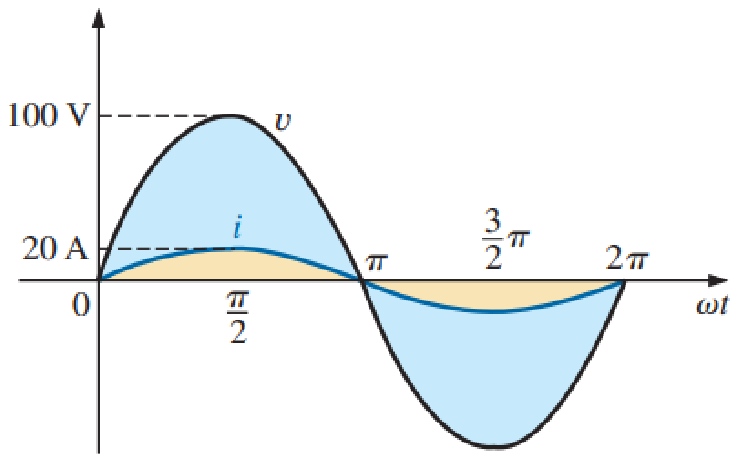

FIG.3 Waveforms for Example 1.

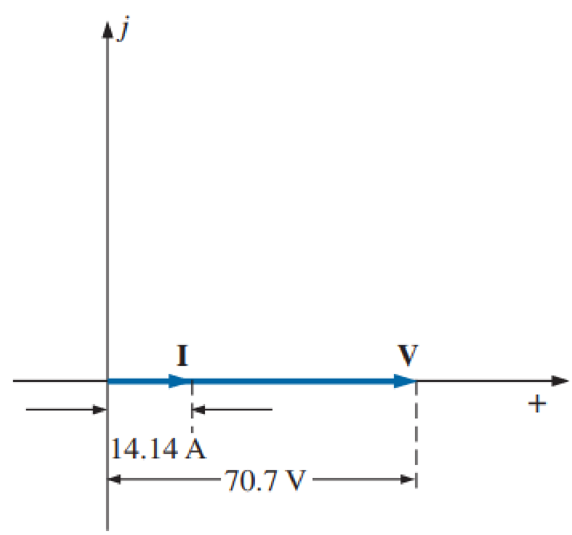

FIG. 4 Phasor diagrams for Examples 1

EXAMPLE 2 Using complex algebra, find the voltage v for the circuit in Fig.5. Sketch the waveforms and the phasor diagram of v and i .

Solution: Note Fig. 6 and Fig. 7:

$$i = 4\sin(\omega t + 30°) \Rightarrow \text{phasor form } \mathbf{I} = 2.828\text{ A} \angle 30°$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_R = (I\angle\theta)(R\angle 0°) = (2.828\text{ A}\angle 30°)(2\ \Omega\angle 0°)$$
$$= 5.656\text{ V}\angle 30°$$

and $$v = \sqrt{2}\,(5.656)\sin(\omega t + 30°) = \mathbf{8.0\sin(\omega t + 30°)}$$

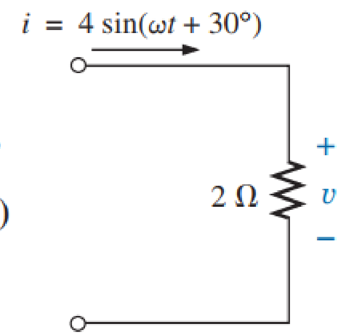

FIG. 5 Example 2.

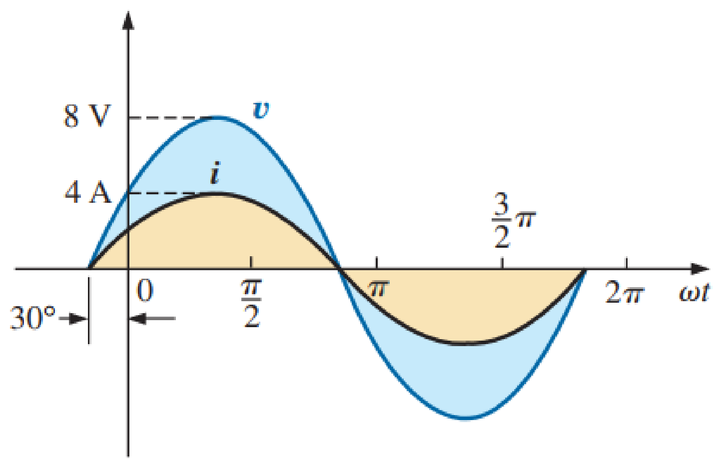

FIG.6 Waveforms for Example 2.

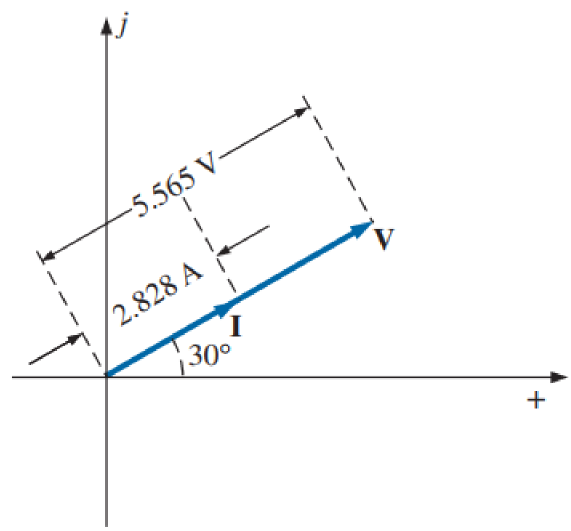

FIG. 7 Phasor diagrams for Example 2

Inductive Reactance

For the pure inductor in Fig.8, the voltage leads the current by 90° and that the reactance of the coil X_L is determined by ωL.

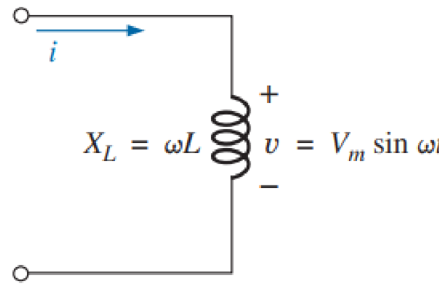

FIG. 8 Inductive ac circuit.

$$v = V_m \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = V\angle 0°$$

By Ohm's law,

$$\mathbf{I} = \frac{V\angle 0°}{X_L\angle\theta_L} = \frac{V}{X_L}\angle 0° - \theta_L$$

Since v leads i by 90°, i must have an angle of 90° associated with it. To satisfy this condition, θ_L must equal +90°. Substituting θ_L =90°, we obtain

$$\mathbf{I} = \frac{V\angle 0°}{X_L\angle 90°} = \frac{V}{X_L}\angle 0° - 90° = \frac{V}{X_L}\angle -90°$$

so that in the time domain,

$$i = \sqrt{2}\left(\frac{V}{X_L}\right)\sin(\omega t - 90°)$$

We use the fact that θ_L= 90° in the following polar format for inductive reactance to ensure the proper phase relationship between the voltage and current of an inductor:

$$\boxed{\mathbf{Z}_L = X_L\angle 90°}$$

EXAMPLE 3 Using complex algebra, find the current i for the circuit in Fig. 9. Sketch the v and i curves and phasor diagram.

Solution: Note Fig. 10 and Fig.11:

$$v = 24 \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = 16.968 \text{ V}\angle 0°$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_L} = \frac{V\angle\theta}{X_L\angle 90°} = \frac{16.968 \text{ V}\angle 0°}{3\ \Omega\angle 90°} = 5.656 \text{ A}\angle -90°$$

and $$i = \sqrt{2}(5.656)\sin(\omega t - 90°) = \mathbf{8.0 \sin(\omega t - 90°)}$$

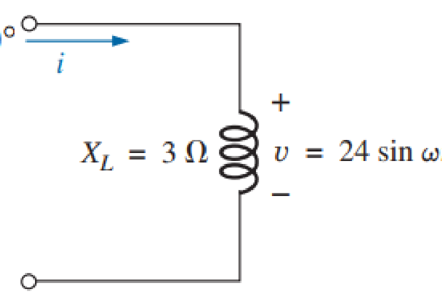

FIG. 9 Example 3.

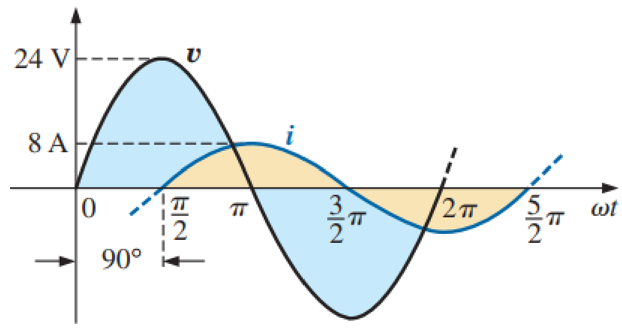

FIG.10 Waveforms for Example 3.

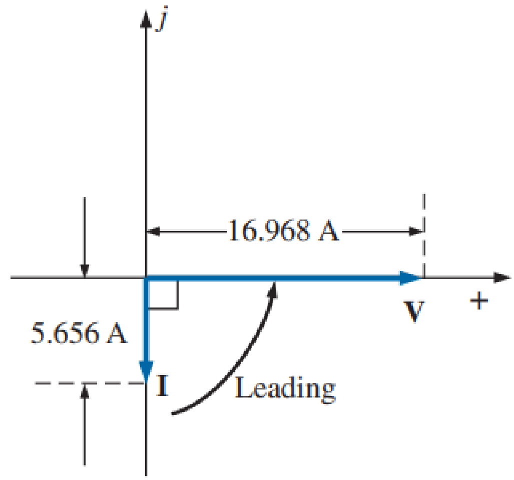

FIG. 11 Phasor diagrams for Example 3

EXAMPLE 4 Using complex algebra, find the voltage v for the circuit in Fig. 12. Sketch the v and i curves and phasor diagram.

Solution: Note Fig. 13 and Fig. 14:

$$i = 5 \sin(\omega t + 30°) \Rightarrow \text{phasor form } \mathbf{I} = 3.535 \text{ A} \angle 30°$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_L = (I \angle \theta)(X_L \angle 90°) = (3.535 \text{ A} \angle 30°)(4\ \Omega \angle +90°)$$
$$= 14.140 \text{ V} \angle 120°$$

and $$v = \sqrt{2}(14.140) \sin(\omega t + 120°) = \mathbf{20 \sin(\omega t + 120°)}$$

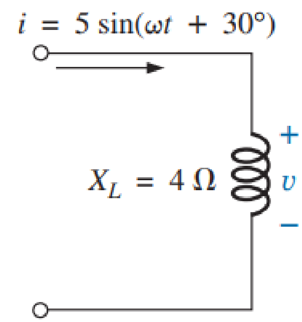

FIG. 12 Example 4.

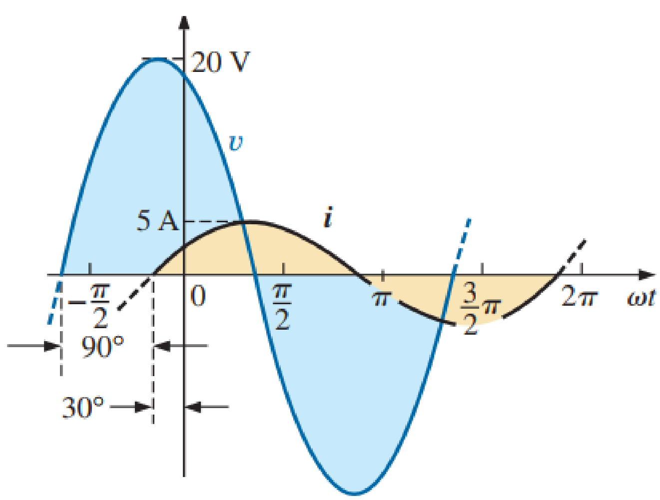

FIG.13 Waveforms for Example 4.

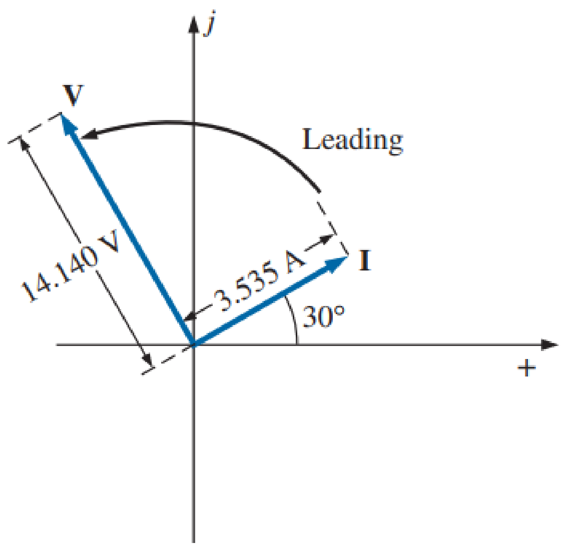

FIG. 14 Phasor diagrams for Example 4

Capacitive Reactance

for the pure capacitor in Fig. 15, the current leads the voltage by 90° and that the reactance of the capacitor X_C is determined by $1/\omega C$.

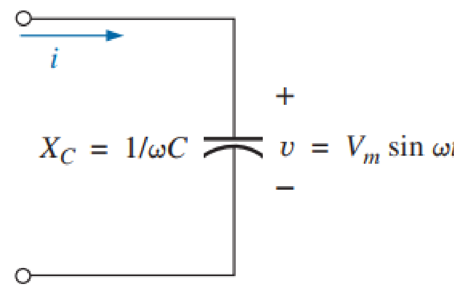

FIG. 15 Capacitive ac circuit.

$\upsilon = V_m \sin \omega t \Rightarrow$ phasor form $\mathbf{V} = V \angle 0°$

Applying Ohm's law and using phasor algebra, we find

$$\mathbf{I} = \frac{V \angle 0°}{X_C \angle \theta_C} = \frac{V}{X_C} \angle 0° - \theta_C$$

Since i leads y by 90°, i must have an angle of 90° associated with it. To satisfy this condition, θ_C must equal 90°. Substituting θ_C = 90° yields

$$\mathbf{I} = \frac{V \angle 0°}{X_C \angle -90°} = \frac{V}{X_C} \angle 0° - (-90°) = \frac{V}{X_C} \angle 90°$$

so, in the time domain,

$$i = \sqrt{2}\left(\frac{V}{X_C}\right) \sin(\omega t + 90°)$$

We use the fact that θ_C = 90° in the following polar format for capacitive reactance to ensure the proper phase relationship between the voltage and current of a capacitor:

$$\boxed{\mathbf{Z}_C = X_C \ \angle -90°}$$

EXAMPLE 5 Using complex algebra, find the current i for the circuit in Fig. 16. Sketch the v and i curves and phasor diagram.

Solution: Note Fig. 17 and Fig. 18:

$$v = 15 \sin \omega t \Rightarrow \text{phasor notation } \mathbf{V} = 10.605 \text{ V} \angle 0^\circ$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_C} = \frac{V \angle \theta}{X_C \angle -90^\circ} = \frac{10.605 \text{ V} \angle 0^\circ}{2 \,\Omega \angle -90^\circ} = 5.303 \text{ A} \angle 90^\circ$$

and

$$i = \sqrt{2}(5.303) \sin(\omega t + 90^\circ) = \mathbf{7.5 \sin(\omega t + 90^\circ)}$$

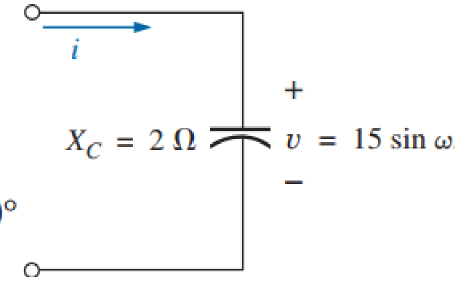

FIG. 16 Example 5.

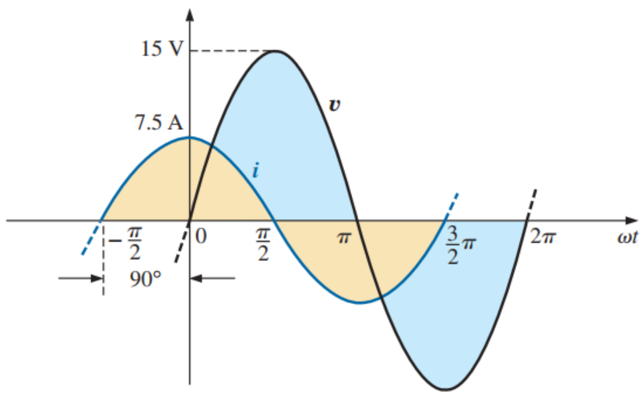

FIG.17 Waveforms for Example 5.

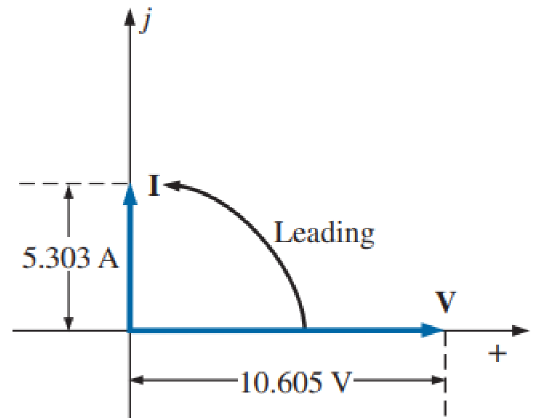

FIG. 18 Phasor diagrams for Example 5.

EXAMPLE 6 Using complex algebra, find the voltage v for the circuit in Fig. 19. Sketch the v and i curves and phasor diagram.

Solution: Note Fig. 20 and Fig. 21:

$$i = 6 \sin(\omega t - 60^\circ) \Rightarrow \text{phasor notation } \mathbf{I} = 4.242 \text{ A} \angle -60^\circ$$

$$\begin{aligned} \mathbf{V} &= \mathbf{I}\mathbf{Z}_C = (I \angle \theta)(X_C \angle -90^\circ) = (4.242 \text{ A} \angle -60^\circ)(0.5 \,\Omega \angle -90^\circ) \\ &= 2.121 \text{ V} \angle -150^\circ \end{aligned}$$

and

$$v = \sqrt{2}(2.121) \sin(\omega t - 150^\circ) = \mathbf{3.0 \sin(\omega t - 150^\circ)}$$

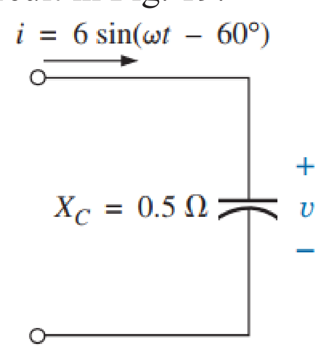

FIG. 19 Example 6.

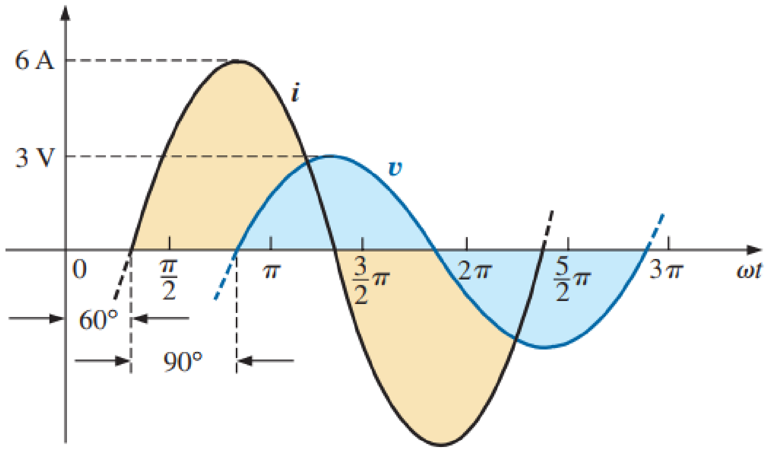

FIG.20 Waveforms for Example 6.

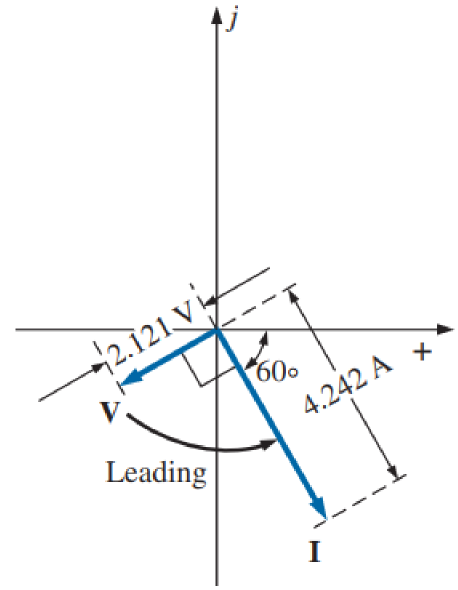

FIG. 21 Phasor diagrams for Example 6.

Impedance Diagram

Now that an angle is associated with resistance, inductive reactance, and capacitive reactance, each can be placed on a complex plane diagram, as shown in Fig. 22. For any network, the resistance will *always* appear on the positive real axis, the inductive reactance on the positive imaginary axis, and the capacitive reactance on the negative imaginary axis.

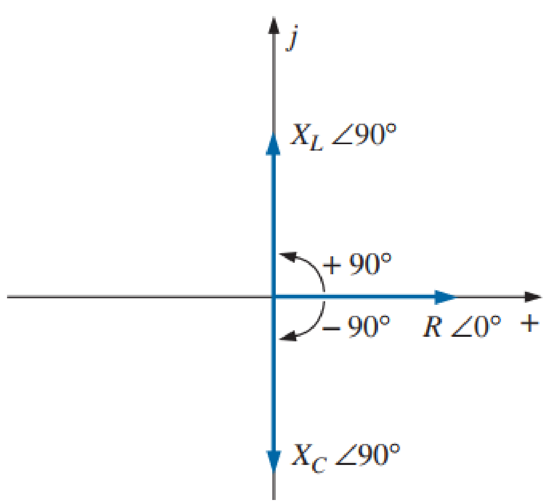

FIG. 22 Impedance diagram

For any configuration (series, parallel, series-parallel, and so on), the angle associated with the total impedance is the angle by which the applied voltage leads the source current. For inductive networks, θ_T will be positive, whereas for capacitive networks, θ_T will be negative.

SERIES CONFIGURATION

The overall properties of series ac circuits (Fig. 23) are the same as those for dc circuits. For instance, the total impedance of a system is the sum of the individual impedances:

$$\boxed{\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \cdots + \mathbf{Z}_N}$$

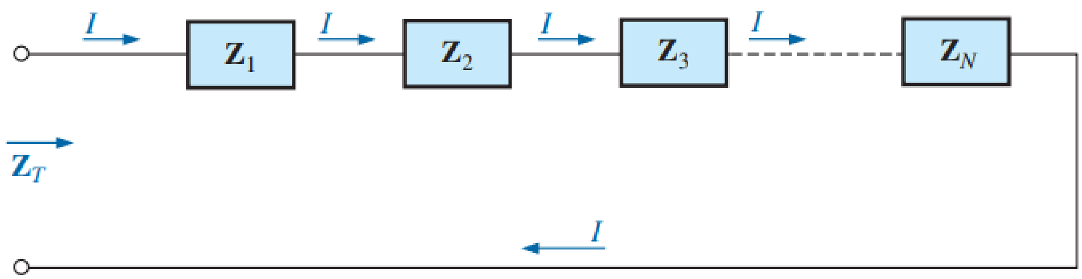

FIG. 23 Series impedances.

EXAMPLE 7 Determine the input impedance to the series network in Fig. 24. Draw the impedance diagram.

Solution:

$$\begin{aligned}\mathbf{Z}_T &= \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3\\ &= R\angle 0^\circ + X_L\angle 90^\circ + X_C\angle -90^\circ\\ &= R + jX_L - jX_C\\ &= R + j(X_L - X_C) = 6\,\Omega + j(10\,\Omega - 12\,\Omega) = 6\,\Omega - j2\,\Omega\end{aligned}$$

$$\mathbf{Z}_T = \mathbf{6.32\,\Omega\angle -18.43^\circ}$$

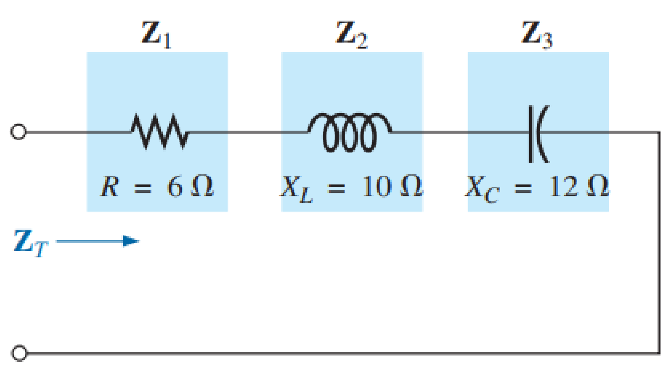

FIG. 24 Example 7.

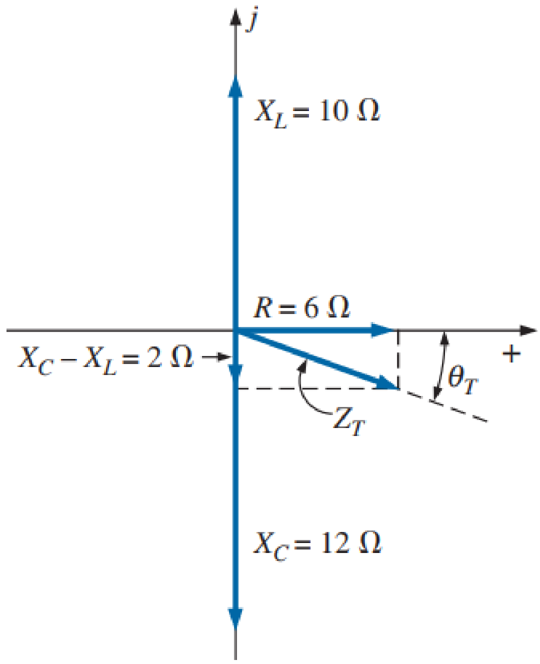

FIG. 25 Impedance diagram for Example 7

For the representative **series ac configuration** in Fig. 26 having two impedances, the current is the same through each element (as it was for the series dc circuits) and is determined by Ohm's law:

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2$$

and

$$\boxed{\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T}}$$

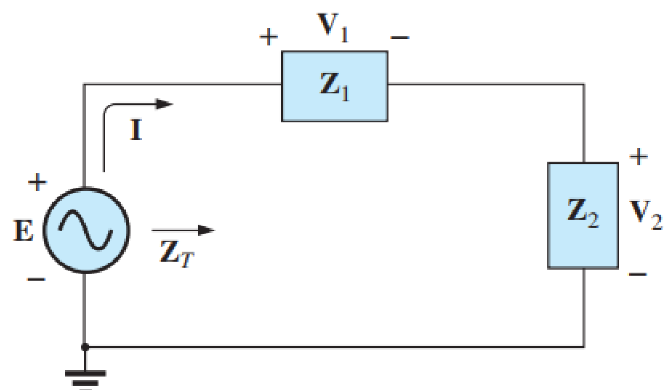

FIG. 26 Series ac circuit.

The voltage across each element can then be found by another application of Ohm's law:

$$\boxed{\mathbf{V}_1 = \mathbf{I}\mathbf{Z}_1}$$

$$\boxed{\mathbf{V}_2 = \mathbf{I}\mathbf{Z}_2}$$

Kirchhoff's voltage law can then be applied in the same manner as it is employed for dc circuits. However, keep in mind that we are now dealing with the algebraic manipulation of quantities that have both magnitude and direction.

$$\mathbf{E} - \mathbf{V}_1 - \mathbf{V}_2 = 0$$

or

$$\boxed{\mathbf{E} = \mathbf{V}_1 + \mathbf{V}_2}$$

The power to the circuit can be determined by

$$\boxed{P = EI\cos\theta_T}$$

where θ_T is the phase angle between $\boldsymbol{E}$ and $\boldsymbol{I}$.

If we write the basic power equation $p = EI\cos\theta$ as follows:

$$\cos\theta = \frac{P}{EI}$$

where $\boldsymbol{E}$ and $\boldsymbol{I}$ are the input quantities and $\boldsymbol{P}$ is the power delivered to the network, and then perform the following substitutions from the basic series ac circuit:

$$\cos\theta = \frac{P}{EI} = \frac{I^2R}{EI} = \frac{IR}{E} = \frac{R}{E/I} = \frac{R}{Z_T}$$

we find the power factor is:

$$\boxed{F_p = \cos\theta_T = \frac{R}{Z_T}}$$

VOLTAGE DIVIDER RULE

The basic format for the **voltage divider rule** in ac circuits is exactly the same as that for dc circuits:

$$\boxed{\mathbf{V}_x = \frac{\mathbf{Z}_x\mathbf{E}}{\mathbf{Z}_T}}$$

where $\mathbf{V}_x$ is the voltage across one or more elements in a series that have total impedance $\mathbf{Z}_x$, $\boldsymbol{E}$ is the total voltage appearing across the series circuit, and $\mathbf{Z}_T$ is the total impedance of the series circuit.

EXAMPLE 8 Using the voltage divider rule, find the voltage across each element of the circuit in Fig. 27.

Solution:

$$\mathbf{V}_C = \frac{\mathbf{Z}_C\mathbf{E}}{\mathbf{Z}_C + \mathbf{Z}_R} = \frac{(4\ \Omega \angle -90°)(100\ \text{V} \angle 0°)}{4\ \Omega \angle -90° + 3\ \Omega \angle 0°} = \frac{400 \angle -90°}{3 - j4}$$

$$= \frac{400 \angle -90°}{5 \angle -53.13°} = \mathbf{80\ V \angle -36.87°}$$

$$\mathbf{V}_R = \frac{\mathbf{Z}_R\mathbf{E}}{\mathbf{Z}_C + \mathbf{Z}_R} = \frac{(3\ \Omega \angle 0°)(100\ \text{V} \angle 0°)}{5\ \Omega \angle -53.13°} = \frac{300 \angle 0°}{5 \angle -53.13°}$$

$$= \mathbf{60\ V \angle +53.13°}$$

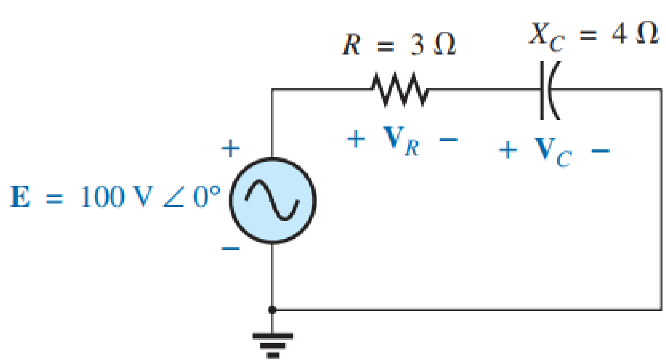

FIG. 27 Example 8.

EXAMPLE 9 Using the voltage divider rule, find the unknown voltages $\mathbf{V}_R$, $\mathbf{V}_L$, $\mathbf{V}_C$, and $\mathbf{V}_1$ for the circuit in Fig. 28.

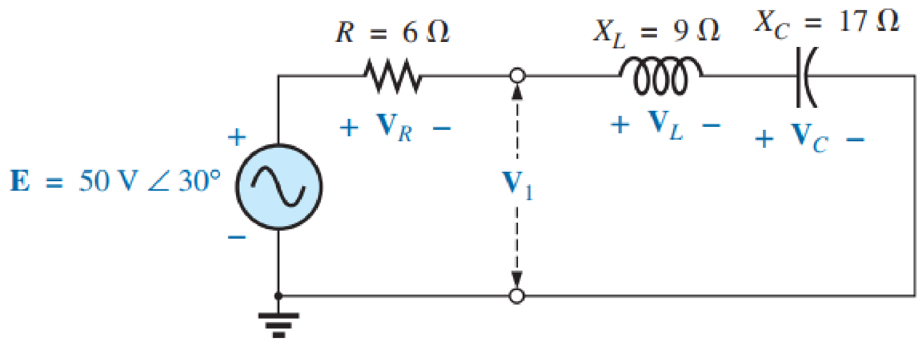

FIG. 28 Example 9.

Solution:

$$\mathbf{V}_R = \frac{\mathbf{Z}_R\mathbf{E}}{\mathbf{Z}_R + \mathbf{Z}_L + \mathbf{Z}_C} = \frac{(6\ \Omega \angle 0°)(50\ \text{V} \angle 30°)}{6\ \Omega \angle 0° + 9\ \Omega \angle 90° + 17\ \Omega \angle -90°}$$

$$= \frac{300 \angle 30°}{6 + j\,9 - j\,17} = \frac{300 \angle 30°}{6 - j\,8}$$

$$= \frac{300 \angle 30°}{10 \angle -53.13°} = \mathbf{30\ V \angle 83.13°}$$

$$\mathbf{V}_L = \frac{\mathbf{Z}_L\mathbf{E}}{\mathbf{Z}_T} = \frac{(9\ \Omega \angle 90°)(50\ \text{V} \angle 30°)}{10\ \Omega \angle -53.13°} = \frac{450\ \text{V} \angle 120°}{10 \angle -53.13°}$$

$$= \mathbf{45\ V \angle 173.13°}$$

$$\mathbf{V}_C = \frac{\mathbf{Z}_C\mathbf{E}}{\mathbf{Z}_T} = \frac{(17\ \Omega \angle -90°)(50\ \text{V} \angle 30°)}{10\ \Omega \angle -53.13°} = \frac{850\ \text{V} \angle -60°}{10 \angle -53°}$$

$$= \mathbf{85\ V \angle\ -6.87°}$$

$$\mathbf{V}_1 = \frac{(\mathbf{Z}_L + \mathbf{Z}_C)\mathbf{E}}{\mathbf{Z}_T} = \frac{(9\ \Omega \angle 90° + 17\ \Omega \angle -90°)(50\ \text{V} \angle 30°)}{10\ \Omega \angle -53.13°}$$

$$= \frac{(8 \angle -90°)(50 \angle 30°)}{10 \angle -53.13°}$$

$$= \frac{400 \angle -60°}{10 \angle -53.13°} = \mathbf{40\ V \angle -6.87°}$$