Bilad Alrafidain University College

Electric Power Techniques Engineering Department

Control Systems Analysis

Fourth Stage

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Lecture Twelve

Root Locus Method

Assistant Lecturer. Ibrahim Ismail

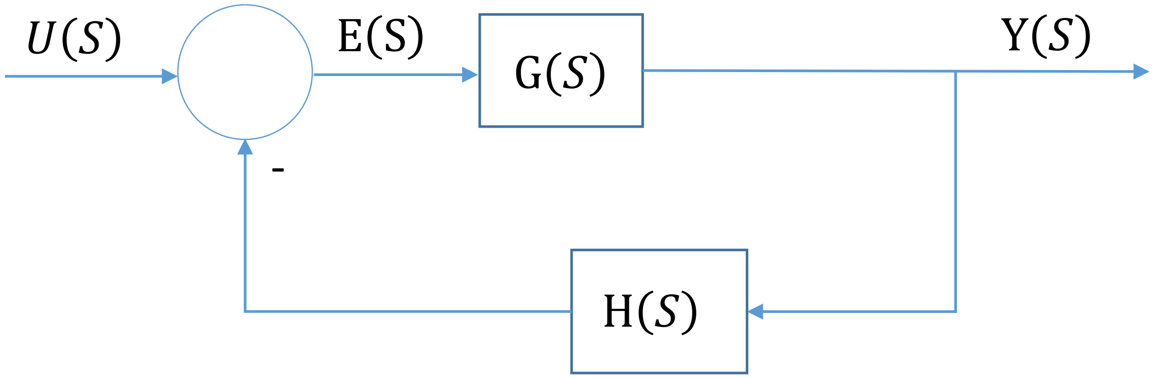
Root Locus Method

The **root locus** is a graphical representation in s-domain and it is symmetrical about the real axis. Because the open loop poles and zeros exist in the s-domain having the values either as real or as complex conjugate pairs.

Root Locus Method

Example 1: A simplified form of the open-loop transfer function of an airplane with an autopilot in the longitudinal mode is:
a) Sketch the root-locus plot for the system.
b) Find the range of gain *K* for closed loop system stability

$$G(S)H(S) = \frac{K(S+1)}{S(S-1)(S^2+4S+16)}$$



$$\frac{Y(S)}{U(S)} = \frac{G(S)}{1+G(S)H(S)}$$ Transfer function

$$P(S) = 1 + G(S)H(S) = 0$$ Characteristic equation

$$G(S)H(S) = -1$$

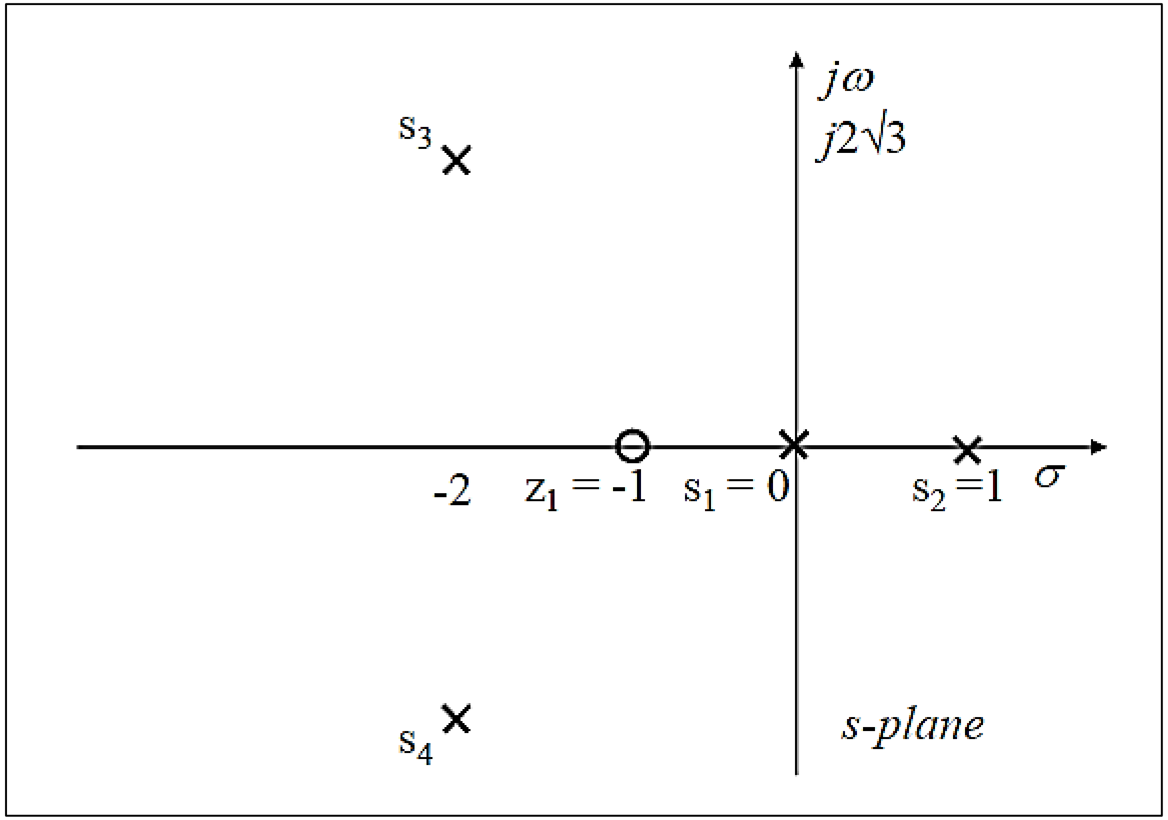
$$G(S)H(S) = \frac{K(S+1)}{S(S-1)(S^2+4S+16)}$$ Open Loop Transfer function

Root Locus Method

1. Locate the poles and zeros of $G(s)H(s)$ on the s plane: The first step in constructing a root-locus plot is to locate the open-loop poles $G(s)H(s)$;

$S_1 = 0, S_2 = 1 \; and \; S_{3,4} = -2 \pm j2\sqrt{3}$

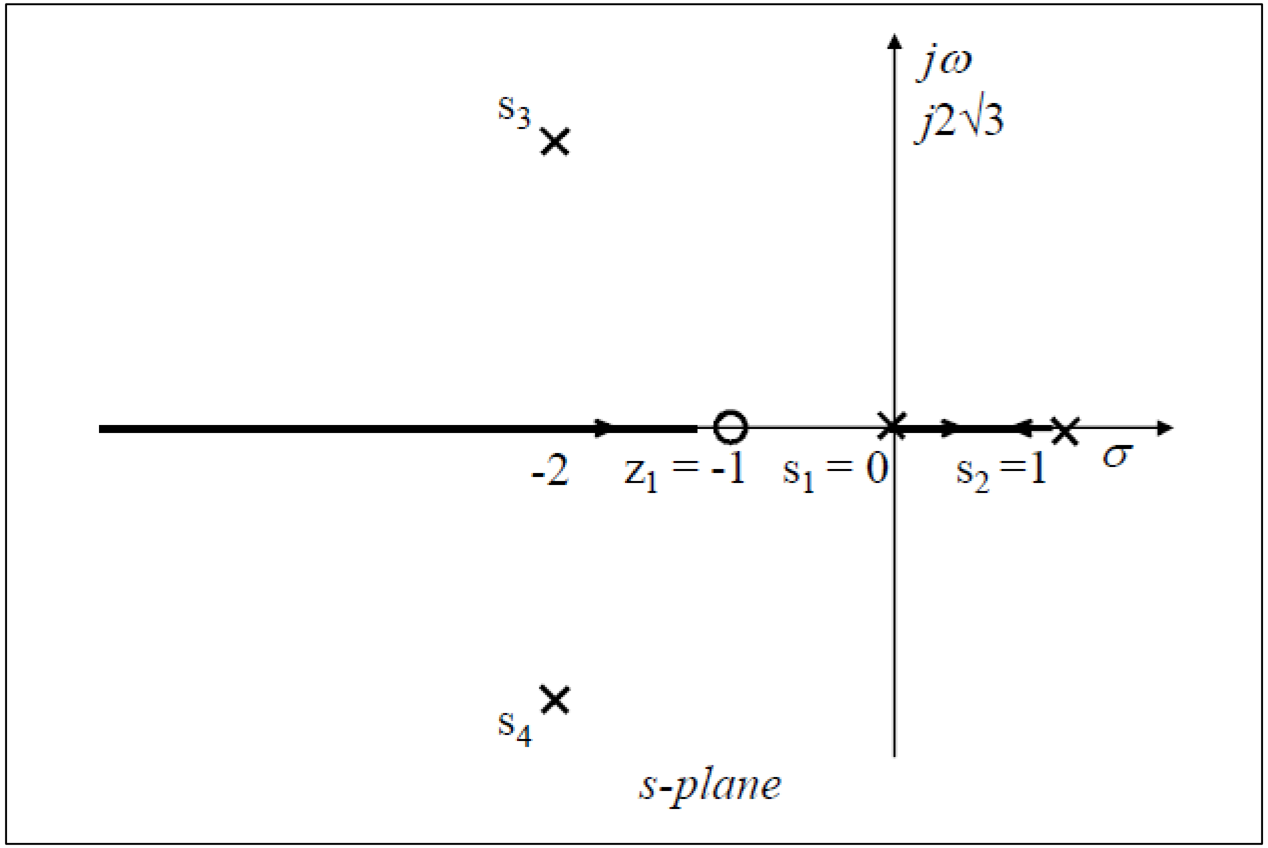
The open-loop poles $G(s)H(s)$; $Z_1 = -1$



Root Locus Method

2. Determine the root loci on the real axis: Root loci exist on the real axis between 1 and 0 and between -1 and -∞.

$$G(S)H(S) = \frac{K(S+1)}{S(S-1)(S^2+4S+16)}$$



Root Locus Method

3. Determine the asymptotes of root loci: Since n=4 and m=1 there are three asymptotes whose angles can be determined as

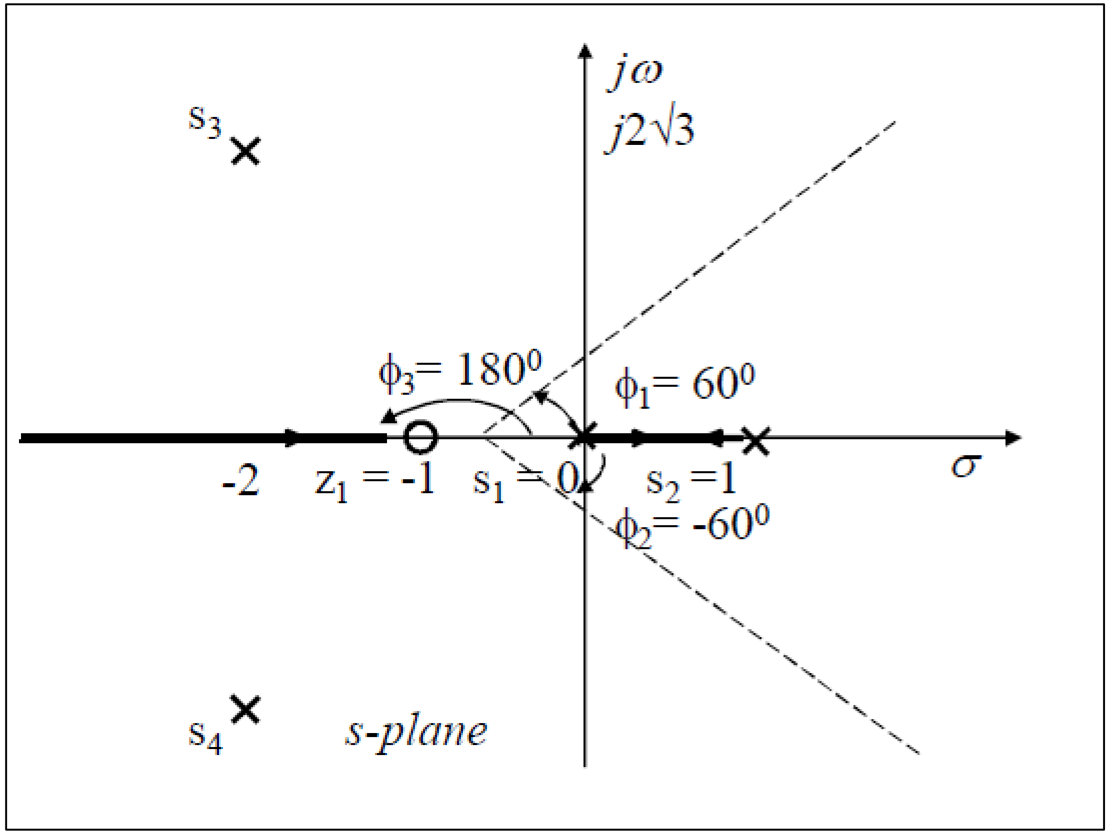
The angle of asymptote: $\emptyset_j = \dfrac{\pm 180^{\circ}(2K+1)}{n-m}, K = 0,1,2,\ldots$

For $k = 0 : \emptyset_1 = 60°, and\ \emptyset_2 = -60°$
For $k = 1\ \emptyset_3 = \pm 180°$

All the asymptotes intersect on the real axis at:

$$\sigma = -\frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m}$$

$$\sigma = -\frac{\left(0 - 1 + 2 + j2\sqrt{3} + 2 - j2\sqrt{3}\right) - (1)}{4-1} = -\frac{2}{3}$$



Root Locus Method

4. Find the breakaway and break-in points : Depending on the characteristic equation P(s).

The characteristic equation is given by : $P(S) = 1 + K\frac{(S+1)}{S(S-1)(S^2+4S+16)} = 0$

$$K = -\frac{S(S-1)(S^2+4S+16)}{(S+1)}$$

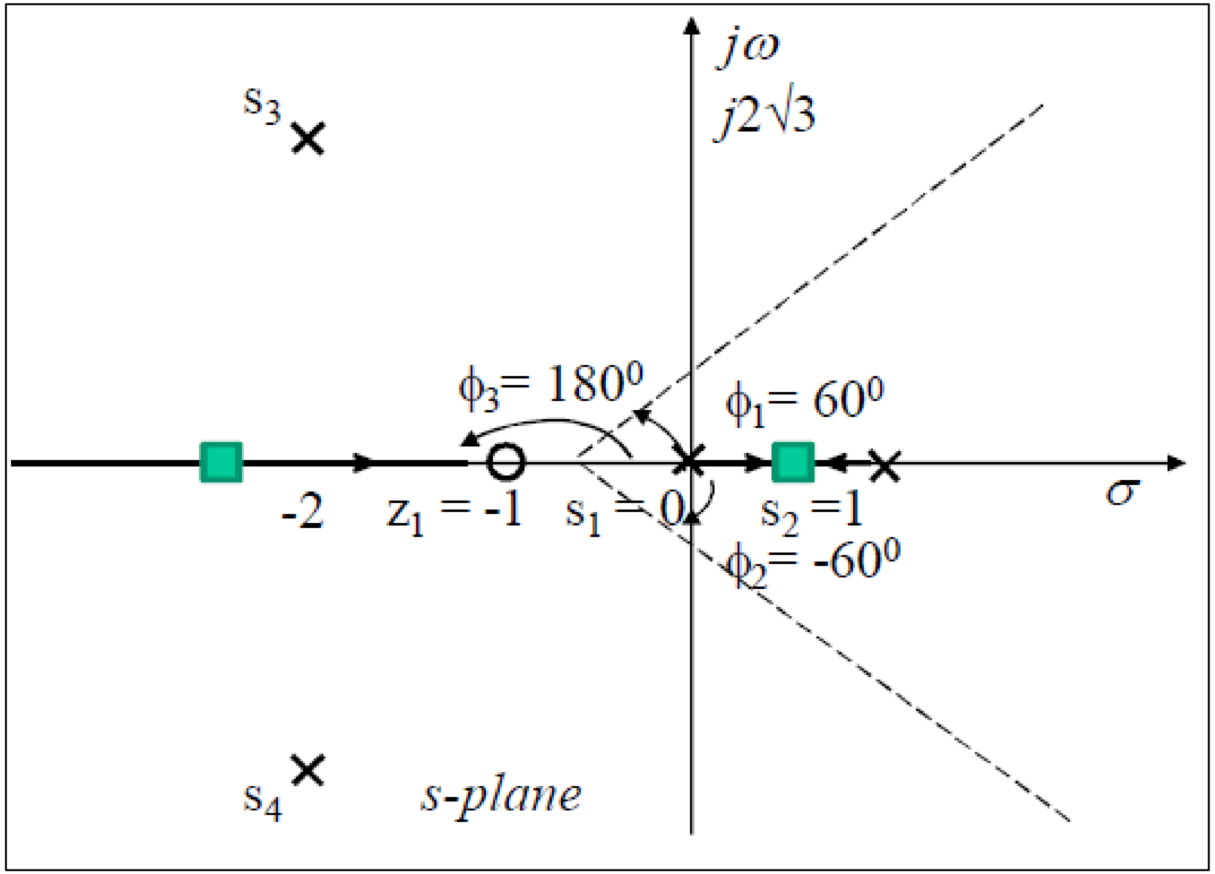
By differentiating K with respect to S, we get :

$$\frac{dK}{dS} = -\frac{3S^4+10S^3+21S^2+24S-16}{(S+1)^2} = 0 \Rightarrow \begin{cases} S_1 = 0.45 \\ S_2 = -2.26 \\ S_{3,4} = -0.76 \pm j2.16 \end{cases}$$

Points $S_1 = 0.45$ and $S_2 = -2.26$ are on root loci on the real axis. Hence, these points are actual breakaway and break-in points, respectively. Points $S_{3,4} = -0.76 \pm j2.16$ do not satisfy the angle condition. Hence, they are neither breakaway nor break-in points.

Root Locus Method

$S_1 = 0.45$ is breakaway point and $S_2 = -2.26$ break-in point.



Root Locus Method

5. Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero).

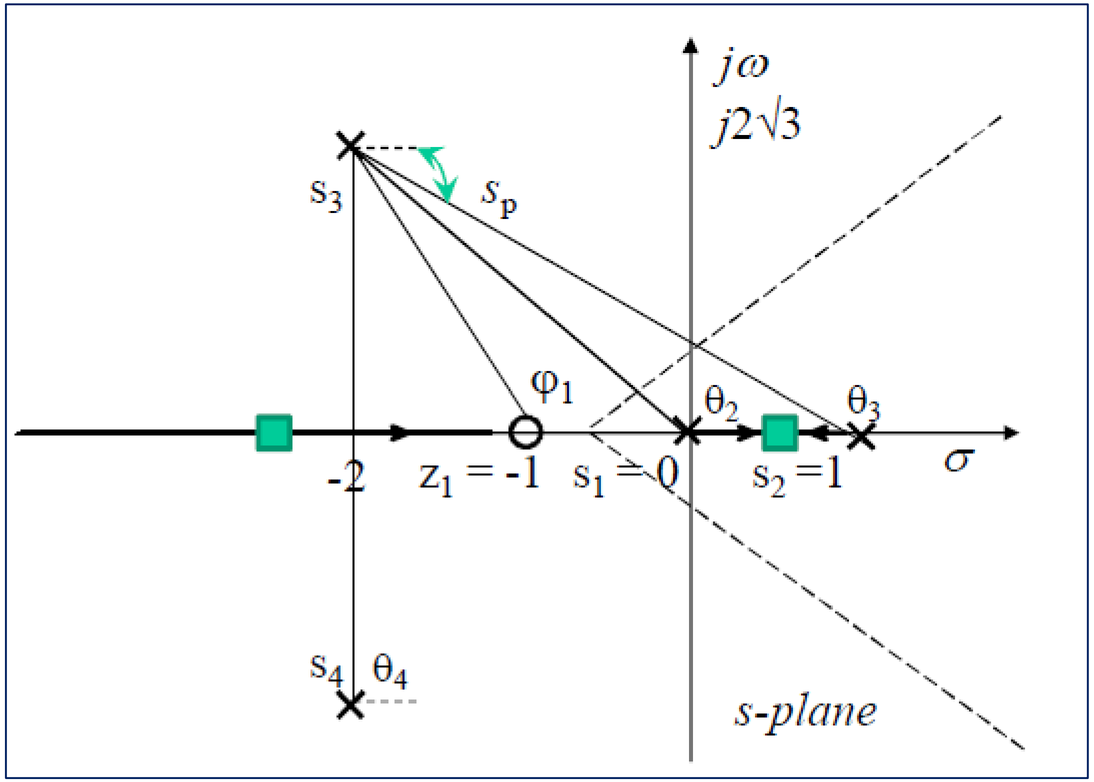
For the open-loop poles $G(s)H(s)$ at $S_{3,4} = -2 \pm j2\sqrt{3}$, the angle of departure θ is

Angle of departure from a complex pole

$$\theta = 180^\circ - \sum_{j=1}^{n-1} \theta_j + \sum_{i=1}^{m} \varphi_i$$

$$\theta = 180^\circ - 120^\circ - 130.5^\circ - 90^\circ + 106^\circ$$

$$\theta = -54.5^\circ$$



Root Locus Method

6. Find the points where the root loci may cross the imaginary axis: letting $s = j\omega$ in the characteristic equation, equating both the real part and the imaginary part to zero, and solving for ω and K.

$$P(S) = 1 + K\frac{(S+1)}{S(S-1)(S^2+4S+16)} = 0$$

$$P(S) = S^4 + 3S^3 + 12S^2 + KS - 16S + K = 0$$

letting $s = j\omega$ in the characteristic equation

$$P(S)\Big|_{s=j\omega} = 1 + G(j\omega)H(j\omega) = 0$$

$$(j\omega)^4 + 3(j\omega)^3 + 12(j\omega)^2 + K(j\omega) - 16(j\omega) + K = 0$$

$$\omega^4 - 3j\omega^3 - 12\omega^2 + Kj\omega - 16j\omega + K = 0$$

Equating the Imaginary parts which contains j with zero, then Equating the Real Parts which not contains j with zero and solving for ω and K.

Root Locus Method

$$\omega^4 - 3j\omega^3 - 12\omega^2 + Kj\omega - 16j\omega + K = 0$$

$$-3j\omega^3 + (K-16)j\omega = 0 \Rightarrow 3j\omega^3 = (K-16)j\omega$$

$$3\omega^2 = K - 16 \Rightarrow \omega^2 = \frac{K-16}{3} \quad \dots (1)$$

$\omega^4 - 12\omega^2 + K = 0$ Then $\omega^4 - 12\left(\frac{K-16}{3}\right) + K = 0 \Rightarrow \omega^4 - 4K + 64 + K = 0 \Rightarrow \omega^4 + 64 - 3K = 0$

$$\omega^4 = 3K - 64 \Rightarrow \omega^2 = \sqrt{3K-64} \quad \dots (2)$$

Now, let's equating the above equations (1) and (2) with each other.

$\frac{K-16}{3} = \sqrt{3K-64} \Rightarrow 3K - 64 = \left(\frac{K-16}{3}\right)^2$ Then $3K - 64 = \frac{K^2 - 32K + 256}{9}$

$27K - 576 = K^2 - 32K + 256 \Rightarrow K^2 - 59K + 832 = 0$ Then $K_1 = 35.7$ & $K_2 = 23.3$

The crossing points on the imaginary axis are thus :

For $K_1 = 35.7$ & $K_2 = 23.3$ from equation (1) we find that $\omega_1 = \pm j2.56$ & $\omega_2 = \pm j1.56$

Root Locus Method

From step 6, the system is stable for $23.3 < K < 35.7$. Otherwise, it is unstable. Thus, the system is conditionally stable.